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Distance-Preserving Subgraphs of Hypercubes

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We give a characterization of connected subgraphs G of hypercubes H such that the distance in G between any two vertices $a, b \in G$ is the same as their distance in H. The hypercubes are graphs which generalize the ordinary cube graph.

In this paper we consider graphs which have neither loops nor multiple edges. If G is a graph we shall denote the set of vertices of that graph also by G. If G_1 and G_2 are graphs then a map $f: G_1 \to G_2$ is called *a homomorphism* if $f(a), f(b) \in G_2$ are adjacent whenever $a, b \in G_1$ are adjacent. A homomorphism $f: G_1 \to G_2$ is *full* if the converse of the above statement also holds, i.e., if $a, b \in G_1$ are adjacent whenever $f(a), f(b) \in G_2$ are adjacent. A subgraph G_1 of G_2 is *full* if the inclusion homomorphism $G_1 \to G_2$ is full. Let d_i be the distance function on G_i $(i = 1, 2)$. A homomorphism $f: G_1 \to G_2$ is *distance-preserving* (d.p.) if $d_2(f(a), f(b)) = d_1(a, b)$ for all $a, b \in G_1$. It is clear that a d.p. homomorphism is injective. A connected subgraph G_1 of G_2 is a *d.p. subgraph* if the inclusion homomorphism $G_1 \to G_2$ is a d.p. homomorphism. It is evident that every d.p. subgraph is full.

Let G be a connected graph and d its distance function. A subset $V \subset G$ is *closed* if $a, b \in V$; $x \in G$; $d(a, x) + d(x, b) = d(a, b)$ imply $x \in V$. The intersection of a family of closed subsets of G is also closed. Thus for every subset U of G there exists the unique minimal closed subset of G containing U. We denote this subset by $\overline{U}$ and call it the *closure* of U in G.

If $a, b \in G$ are adjacent we define

$$G(a, b) = \{x \in G \mid d(x, a) < d(x, b)\}.$$

Then $G(a, b)$ and $G(b, a)$ are disjoint and non-empty subsets of G. If G is also bipartite then $G(a, b) \cup G(b, a) = G$.

Now let S be a set and define a graph $H(S)$ as follows: the vertices are the finite subsets of S; two vertices F_1 and F_2 are adjacent if and only if the symmetric difference $F_1 \Delta F_2$ is a singleton set. We shall say that $H(S)$ is *the hypercube on* S.

The object of this paper is to characterize the d.p. subgraphs of hypercubes. If G is a d.p. subgraph of some hypercube $H(S)$ then it is clear that the following holds:

(C_1) G is a connected bipartite graph,

(C_2) If $a, b \in G$ are adjacent then $G(a, b)$ is a closed subset of G.

We shall show that the converse is also valid.

THEOREM 1. *For a graph G the following two assertions are equivalent:*

(i) *G is isomorphic to a d.p. subgraph of a hypercube.*

(ii) *G satisfies (C_1) and (C_2).*

Proof. We have already remarked that (i) implies (ii). Let G be a graph satisfying (C_1) and (C_2). We fix a vertex $c \in G$. We define a binary relation θ on the set E of edges of G. Let $e_1, e_2 \in E$ and let a, b be the vertices of G joined by e_1. We say that $e_1 \,\theta\, e_2$ if and only if e_2 joins a vertex in $G(a, b)$ to a vertex in $G(b, a)$. It is clear that θ is reflexive. Assume $e_1 \,\theta\, e_2$ and let $u \in G(a, b)$, $v \in G(b, a)$ be the vertices joined by e_2. We claim that $G(u, v) = G(a, b)$. It suffices to show that $G(a, b) \subset G(u, v)$. Assume that $x \in G(a, b)$ and $d(x, v) < d(x, u)$. Then v is in the closure of $G(a, b)$ which contradicts (C_2). Hence θ is also symmetric and transitive. Let $S = E/\theta$ be the set of equivalence classes and for $e \in E$ let $\bar{e} \in S$ be the equivalence class containing e. For $x \in G$ we define a subset $F(x)$ of S as follows: $\bar{e} \in F(x)$ if and only if x and c do not belong to the same set $G(a, b)$ or $G(b, a)$ where a and b are the vertices joined by e. It is evident that the choice of the representative e is irrelevant. We claim that $F(x)$ is a finite subset of S. Indeed, let $d(x, c) = m$ and let $e_1, e_2, ..., e_m$ be the edges of some path joining x and c. Then it is easy to see that $F(x) = \{\bar{e}_1, \bar{e}_2, ..., \bar{e}_m\}$. Thus we have a map $f\colon G \to H(S)$ which sends $x \in G$ to $F(x) \in H(S)$. Let $e \in E$ be the edge joining the vertices a and b. We claim that $F(a) \Delta F(b) = \{\bar{e}\}$. It is clear that $\bar{e} \in F(a) \Delta F(b)$. Assume that $\bar{e}_1 \in F(a) \Delta F(b)$. If e_1 joins the vertices $u, v \in G$ then this means that a and b belong to different sets $G(u, v)$ and $G(v, u)$. Thus $e_1 \,\theta\, e$, i.e., $\bar{e}_1 = \bar{e}$. This proves that f is a homomorphism. Now, let a and b be two vertices of G, $d(a, b) = m$ and $e_1, ..., e_m$ the edges of some path joining a and b. It is clear that $F(a) \Delta F(b) = \{\bar{e}_1, \bar{e}_2, ..., \bar{e}_m\}$. We claim that the edges $e_1, e_2, ..., e_m$ are pairwise non-equivalent. Indeed, assume that $e_i \,\theta\, e_j$ for some $i < j$. Let u_k be the vertex incident with e_{k-1} and e_k. Then we would have

$d(u_{j+1}, u_i) < d(u_{j+1}, u_{i+1})$, which contradicts the fact that $d(a, b) = m$. Thus the distance between $F(a)$ and $F(b)$ in $H(S)$ is also m. Therefore f is a d.p. homomorphism and the proof is complete.

If T_1 and T_2 are sets of the same cardinality then the hypercubes $H(T_1)$ and $H(T_2)$ are isomorphic. Let T be a graph satisfying the conditions (C_1) and (C_2). Define dim G to be the smallest cardinal α for which there exists a d.p. homomorphism $f\colon G \to H(T)$ with $\operatorname{card}(T) = \alpha$. With the notation introduced in the proof of Theorem 1, we have:

THEOREM 2. *If G is a graph satisfying (C_1) and (C_2) then* $\dim G = \operatorname{card}(E/\theta)$.

Proof. It is clear from the proof of Theorem 1 that it suffices to show that $\operatorname{card}(E/\theta) \leqslant \dim G$. Let G be a set such that $\operatorname{card}(T) = \dim G$ and let $f\colon G \to H(T)$ be a d.p. homomorphism. Let $a, b \in G$ be adjacent and e the edge which joins them. Then $f(a) \mathbin{\Delta} f(b) = \{t\}$ is a singleton subset of T. Since f is a d.p. homomorphism we have

$$G(a, b) = \{x \in G \mid t \notin f(x) \mathbin{\Delta} f(a)\}.$$

This shows that we have a well-defined map $\phi\colon E/\theta \to T$ such that $\phi(\bar{e}) = t$ is defined as above.

Let e_1 be an edge joining the vertices $a, b \in G$ and e_2 an edge joining $u, v \in G$. Assume that, for instance, $u \in G(a, b)$. Then if $\phi(\bar{e}_1) = \phi(\bar{e}_2)$ we have $f(a) \mathbin{\Delta} f(b) = f(u) \mathbin{\Delta} f(v) = \{t\}$ and $t \notin f(a) \mathbin{\Delta} f(u)$. It follows that $t \notin f(b) \mathbin{\Delta} f(v)$, i.e., $v \in G(b, a)$. Thus $\bar{e}_1 = \bar{e}_2$ and ϕ is injective.

Now, let G be a graph and $f\colon G \to H(S)$ a map into a hypercube. The subsets of S of the form $f(a) \mathbin{\Delta} f(b)$ where $a, b \in G$ are adjacent will be called f-blocks. The following result is due to I. F. Blake when G is a finite graph. Our proof, which we give below, is quite different from his.

THEOREM 3. *Let G be a connected bipartite graph, $m \geqslant 1$ a fixed integer, and $f\colon G \to H(S)$ a map such that $d(f(a), f(b)) = m \cdot d(a, b)$ for all $a, b \in G$. Then any two distinct f-blocks are disjoint. In particular, it follows that G satisfies also (C_2).*

Proof. Note first that every f-block has m elements. Let $T = f(a) \mathbin{\Delta} f(b)$ and $U = f(u) \mathbin{\Delta} f(v)$ be two f-blocks where we assume that a and b are adjacent and also u and v. We can assume that $u \in G(a, b)$ and that $d(a, v) = d(a, u) + 1$. Since

$$f(a) \mathbin{\Delta} f(v) \subset (f(a) \mathbin{\Delta} f(u)) \cup U$$

and $f(a) \Delta f(v)$ has m elements more than $f(a) \Delta f(u)$, it follows that

$$f(a) \Delta f(v) = (f(a) \Delta f(u)) \cup U$$

and that this union is disjoint.

On the other hand we have

$$f(b) \Delta f(v) \subset (f(a) \Delta f(u)) \cup T \cup U$$

and $d(b, v) = d(a, v) \pm 1$. If U and T meet then we must have $d(b, v) = d(a, v) - 1 = d(a, u)$. This implies that we have

$$f(b) \Delta f(u) = (f(b) \Delta f(v)) \cup U.$$

Since also

$$f(b) \Delta f(u) = (f(a) \Delta f(u)) \cup T$$

we deduce that

$$U \subset (f(a) \Delta f(u)) \cup T.$$

But U and $f(a) \Delta f(u)$ are disjoint and so $U = T$.

In order to prove the second assertion let $\bar{S}$ be the set of f-blocks and $c \in G$ a fixed vertex. If $x \in G$ then $f(x) \Delta f(c)$ is a union of k distinct blocks where $k = d(x, c)$. By the first part of the proof these blocks are uniquely determined. Let $F(x)$ be the set of k blocks which are contained in $f(x) \Delta f(c)$.

Thus the map $\phi: G \to H(\bar{S})$ which sends $x \in G$ to $F(x)$ is a d.p. homomorphism. Indeed, it suffices to notice that if $a, b \in G$ then $\phi(a) \Delta \phi(b)$ consists of those f-blocks which are contained in $f(a) \Delta f(b)$. The result now follows from Theorem 1.

Remarks. (1) The question of characterizing subgraphs of hypercubes remains open. It is not difficult to see that there exist connected subgraphs of hypercubes which are not isomorphic to any d.p. subgraph of any hypercube. These are those connected subgraphs of hypercubes which do not satisfy (C_2). An example is a graph obtained from a 3-dimensional cube by deleting one edge.

(2) The problems in this paper have their origin in a problem of communication theory [3], but see also [2]. In graph theoretical terms one wants to characterize those connected graphs G for which there exists a map $f: G \to H(S)$ such that $d(f(a), f(b)) = m \cdot d(a, b)$ for $a, b \in G$ where $m \geqslant 1$ is a fixed integer. Note that Theorems 1 and 3 give a complete solution of this problem for bipartite graphs. Moreover the method used in the proof of Theorem 1 shows how one can construct the map f when G

satisfies the condition (C_2). By Theorem 3 we know that we can take $m = 1$. Theorem 2 says that f is the most economical because we get the smallest possible dimension.

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